

DISASSOCIATED INDISCERNIBLES

By

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TABLE OF CONTENTS

ACKNOWLEDGEMENTS	ii
ABSTRACT	1
CHAPTERS	
1 INTRODUCTION	2
2 PRELIMINARIES	4
2.1 Prerequisites	4
2.2 Ultrafilters and Ultrapowers	4
2.3 Prikry Forcing	6
2.4 Iterated Prikry Forcing	9
3 THE FORCING	11
3.1 Definition	11
3.2 Basic Structure	12
3.3 Prikry Property	14
3.4 More Structure	16
3.5 Genericity Criteria	18
4 GETTING TWO NORMAL MEASURES	24
4.1 The Measures	24
4.2 Properties	27
5 ITERATED DISASSOCIATED INDISCERNIBLES	30
5.1 Definition	30
5.2 Basic Structure	31
5.3 Prikry Property	32
5.4 More Structure	34
6 GETTING MANY NORMAL MEASURES	35
REFERENCES	36
BIOGRAPHICAL SKETCH	37

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By ‘measure’, we mean a κ -complete ultrafilter on some cardinal κ . This corresponds to the sets of positive measure for a two-valued but nontranslation invariant measure in the sense of Lebesgue. We study *normal* measures, that is, measures U such that for any regressive ($f(\alpha) < \alpha$ for all α) function $f : \kappa \rightarrow \kappa$ we have that f is constant on some set $X \in U$. We use the technique of *forcing* to increase the number of normal measures on a single cardinal. We introduce a variant of Prikry forcing. After defining a new hierarchy of large cardinals, we get many normal measures in the presence of less consistency strength than was previously known, strictly less than that of a measure of Mitchell order two.

CHAPTER 1 INTRODUCTION

*Logic. The art of thinking and reasoning
in strict accordance with the limitations and incapacities
of the human misunderstanding.*

-Ambrose Bierce

This paper addresses a mathematical problem in the subject of Set Theory. Set Theory is the branch of mathematical logic concerned with infinity. Georg Cantor discovered in 1873 that the infinite can be sensibly quantized and that there are actually different sizes of infinity. In fact, there are infinitely many different sizes of infinity. Although Cantor established many properties of how the infinite is arranged, some fundamental questions about its structure eluded him. Almost a century later, Paul Cohen of the United States discovered that the structure of infinity is plastic; that is, there are many different possible arrangements of the infinite. This research is an exploration of some of them.

In Chapter 2 we discuss some preliminary material from which we draw our inspiration and much of our technique. Cohen developed a technique he called ‘forcing’ to prove his results about the infinite. We describe Karl Prikry’s version of forcing for altering the cofinality of a measurable cardinal and Magidor’s iteration of Prikry’s technique which leaves some of the cardinals measurable. We state some important properties of both, like the so-called Prikry Property, which is essential for showing that cardinals are preserved, and Mathias’ genericity criterion.

Readers familiar with the preliminary material will readily understand the original ideas in this paper. In Chapter 3 we introduce a forcing for what is essentially the union of the range of a function generic for Magidor's iterated Prikry forcing. We prove the Prikry property. We give a genericity criteria. And as an application, we show in Chapter 4 that under certain interesting conditions it forces cardinals with exactly two normal measures. In Chapter 5 we define an iterated version of this forcing and show in Chapter 6 that cardinals with several normal measures are possible in the presence of less consistency strength than was previously known. The djinn is in the *detalles*. Our arguments for the Prikry property and the genericity criteria require a lot of infinitary combinatorics. Our proofs are longer than their analogs. And the measurable cardinals we deal with contain additional subtlety.

Our motivation was the folk question, 'How many normal measures can a measurable cardinal have?', perhaps due to Stanislaw Ulam. Kunen and Paris [4] forced a measurable cardinal κ to have the maximal number, 2^{2^κ} , of normal measures. Relative to a measurable cardinal whose measure concentrates on measurable cardinals, Mitchell [7] proved that that exactly two normal measures are possible. We confirm this statement and similar results while reducing the hypothesis. Although generally believed, it is still unproven that the consistency strength of a measurable cardinal having exactly two normal measures is that of a single measurable cardinal.

CHAPTER 2 PRELIMINARIES

2.1 Prerequisites

We assume the reader is knowledgeable about the basic Set Theory associated with large cardinals and independence proofs. In particular, he or she should be familiar with measurable cardinals and the theory of forcing. The author highly recommends [1], [2], and [3] as references. Since this work is an exploration of an alteration of iterated Prikry forcing, readers that are already familiar with Prikry forcing and its generalizations will be better prepared to proceed than those who are not.

2.2 Ultrafilters and Ultrapowers

Definition 2.2.1. A *measure* is a κ -complete nonprinciple ultrafilter on a cardinal κ .

If we remove the translation-invariance requirement from Lebesgue measure and restrict to two values instead of non-negative real values, then our measures above correspond to the class of sets of measure one.

The main reason that set theorists are interested in measures is that the corresponding ultrapowers (Definition 2.2.2) of absolutely well-founded models of Set Theory are absolutely well-founded. Hence the ultrapower of the class of all sets V is embeddable into itself by Mostowski's collapse isomorphism. This gives a homomorphism of V into a proper subclass of itself.

Definition 2.2.2. Let U be a measure on κ . Consider the equivalence relation on functions $f, g : \kappa \rightarrow \kappa$,

$$f \sim g \text{ iff } \{\eta < \kappa : f(\eta) = g(\eta)\} \in U.$$

Let $\text{Ult}(\mathbb{V}, U)$ be the model whose universe consists of equivalence classes $[f]$ of functions $f : \kappa \rightarrow \kappa$ under the relation $\sim$. Interpret ‘ $\in$ ’ in $\text{Ult}(\mathbb{V}, U)$ as

$$[f] \in [g] \text{ iff } \{\eta < \kappa : f(\eta) \in g(\eta)\} \in U.$$

Let $j_U : \mathbb{V} \rightarrow \mathfrak{M} \approx \text{Ult}(\mathbb{V}, U)$ be the natural map given by $x \mapsto [c_x]$ where c_x is the constant function $c_x(\eta) = x$, and $\mathfrak{M}$ is Mostowski’s transitive isomorph of the ultrapower.

An element of $\text{Ult}(\mathbb{V}, U)$ is a class of functions which are equal ‘almost everywhere’, to use the measure theoretic terminology, modulo U . Elements are included in other elements if their representative functions are pointwise included almost everywhere.

Definition 2.2.3. A measure U on κ is *normal* if every regressive ($\forall \alpha \ f(\alpha) < \alpha$) function $f : \kappa \rightarrow \kappa$ is constant on some set $X \in U$.

The property of normality gives a clear description of which equivalence class of functions represents which set in the ultrapower. In fact, for normal measures, the equivalence class of the identity function $f(\alpha) = \alpha$ represents the cardinal κ .

Definition 2.2.4. Let $(X_\eta)_{\eta < \kappa}$ be a collection of subsets of κ . The *diagonal intersection* of these sets is the set $\Delta_{\eta < \kappa} X_\eta = \{\gamma < \kappa : \forall \eta < \gamma (\gamma \in X_\eta)\}$.

Proposition 2.2.5. Let U be a measure on κ . The following are equivalent.

1. U is normal.
2. $\forall X (X \cap \kappa \in U \iff \kappa \in j_U(X))$.
3. U is closed under diagonal intersection.

Definitions 4.1.1 and 6.3 are modeled after a forcing-syntactic version of property 2 above.

We will use the following combinatorial observation, due to Rowbottom, infinitely often in what follows.

Theorem 2.2.6. *Let U be a normal measure on κ . Let $f : [\kappa]^{<\omega} \rightarrow \eta$, where $\eta < \kappa$. There exists a set $X \in U$ such that f is constant on $[X]^n$ for each $n < \omega$.*

The set X above is said to be *homogeneous* for f .

2.3 Prikry Forcing

Prikry forces an unbounded sequence of type ω through a measurable cardinal in his dissertation [9]. We include his definition, and statements of some basic results here, mainly for comparison with our present work.

Let κ be a measurable cardinal and let U_κ be a normal measure on κ .

Definition 2.3.1. *Prikry forcing, $\mathbb{P}(U_\kappa)$, has as conditions pairs $\langle s, X \rangle$ such that*

1. $s \in [\kappa]^{<\omega}$
2. $X \in U_\kappa$.

A condition *extends* another, $\langle s, X \rangle \leq \langle s', X' \rangle$, iff

1. s is an end extension of s'
2. $X \subset X'$
3. $s \setminus s' \subset X'$.

If G is $\mathbb{P}(U_\kappa)$ generic, let $\mathfrak{S} = \cup\{s : \exists X \in U_\kappa \langle s, X \rangle \in G\}$. The set $\mathfrak{S}$ has order type ω , and a standard density argument proves that it is unbounded in κ . After forcing with $\mathbb{P}(U_\kappa)$ the cardinal κ has cofinality ω . A crucial component in the theory is the following.

Theorem 2.3.2. (Prikry Property) *Let σ be a formula in the language of $\mathbb{P}(U_\kappa)$ and $\langle s, X \rangle$ be any condition. Then there exists a condition $\langle s, X' \rangle \leq \langle s, X \rangle$ that decides σ , that is, such that either $\langle s, X' \rangle \Vdash \sigma$ or $\langle s, X' \rangle \Vdash \neg \sigma$*

We will occasionally use the notation ' $p \parallel \sigma$ ' for ' p decides σ '.

Corollary 2.3.3. *Prikry forcing preserves cardinals.*

Adrian Mathias [6] proved not only that $\mathfrak{S}$ all but finitely included in every $X \in U_\kappa$, but in fact the almost-containment of $\mathfrak{S}$ exactly characterizes the measure one sets:

Theorem 2.3.4. (Mathias' Genericity Criteria) *Let $\mathfrak{S} \subset \kappa$. Then $\mathfrak{S}$ is $\mathbb{P}(U_\kappa)$ generic if and only if for all $X \subset \kappa$,*

$$|\mathfrak{S} \setminus X| < \aleph_0 \iff X \in U_\kappa.$$

We devote the rest of this section to examining some properties of the set $\mathfrak{S}$.

Definition 2.3.5. Let $\mathfrak{M} = \langle M, <, = \rangle$ be a model and let I be a simply ordered subclass of M . Then I consists of *first-order indiscernibles over $\mathfrak{M}$* iff for any formula $\varphi(x_1, \dots, x_n)$ and any sequences $\gamma_1 < \dots < \gamma_n$ and $\gamma'_1 < \dots < \gamma'_n$ from I we have

$$\mathfrak{M} \models \varphi[\gamma_1, \dots, \gamma_n] \iff \mathfrak{M} \models \varphi[\gamma'_1, \dots, \gamma'_n].$$

This next theorem is intended to explain our usage of the term 'indiscernibles' when referring to a Prikry sequence. Kunen noticed the first part, but the second part was noticed either by Solovay, Mathias, or Kunen - the history is a bit cloudy.

Theorem 2.3.6. *Let U_0 be a normal ultrafilter on κ_0 in a model $\mathfrak{M}_0$ of ZFC. Let $i_{n,n+1} : \mathfrak{M}_n \rightarrow \mathfrak{M}_{n+1} \approx \text{Ult}(\mathfrak{M}_n, U_n)$ be the map from $\mathfrak{M}_n$ to the transitive collapse of the ultrapower of $\mathfrak{M}_n$ by U_n . Denote $U_{n+1} = i_{n,n+1}(U_n)$ and $\kappa_{n+1} = i_{n,n+1}(\kappa_n)$. Let $\mathfrak{M}_\omega$ be the transitive collapse of the direct limit of the above system of ultrapowers,*

and let κ_ω and U_ω be the images of κ_0 and U_0 respectively. Finally, denote $\mathfrak{S} = \{\kappa_n : n < \omega\}$. Then:

1. $\mathfrak{S}$ is a set of first-order indiscernibles over $\mathfrak{M}_\omega$.
2. $\mathfrak{S}$ is $\mathbb{P}(U_\omega)$ -generic over $\mathfrak{M}_\omega$.

Although Prikry sequences do not in general consist of first-order indiscernibles in the above sense, this is nearly the case.

Theorem 2.3.7. *Let $\mathfrak{S}$ be $\mathbb{P}(U)$ -generic, where U is a measure on κ . Then for each formula $\varphi(x_1, \dots, x_n)$ there is a $\lambda_\varphi < \kappa$ such that for any sequences $\gamma_1 < \dots < \gamma_n$, $\gamma'_1 < \dots < \gamma'_n$ from $\mathfrak{S} \setminus \lambda_\varphi$, we have*

$$\varphi[\gamma_1, \dots, \gamma_n] \iff \varphi[\gamma'_1, \dots, \gamma'_n].$$

Proof: First, let U^n be the ultrafilter consisting of sets X such that for some $H \in U$ we have $[H]^n \subset X$. Consider the sets

$$T = \{(\gamma_1, \dots, \gamma_n) : \gamma_1 < \dots < \gamma_n < \mu \text{ and } \varphi[\gamma_1, \dots, \gamma_n]\},$$

$$F = \{(\gamma_1, \dots, \gamma_n) : \gamma_1 < \dots < \gamma_n < \mu \text{ and } \neg \varphi[\gamma_1, \dots, \gamma_n]\},$$

where μ is the critical point of U . Since T is the complement of F , one of these sets must be in U^n . Without loss of generality, suppose $T \in U^n$. Thus, for some $H \in U$, we have $[H]^n \subset T$. It follows that the set $\{\langle s, X \rangle : [X]^n \subset T\}$ is dense in $\mathbb{P}(U)$. Hence all but finitely many elements of $\mathfrak{S}$ come from H . The theorem follows. ■

We now show that the above theorem is the best possible. That is, we cannot in general get a Prikry generic sequence consisting of first order indiscernibles over the ground model.

Theorem 2.3.8. *There is a model $\mathfrak{M}$ of “ZFC + there is a measurable cardinal” and a class of parameter-free formulas $\{\varphi_x(v)\}_{x \in \mathfrak{M}}$ such that each φ_x is satisfied by x and only x .*

Proof: Consider the minimal set model $L_\alpha[U]$ of “ZFC + there is a measurable cardinal”. Let $\mathfrak{M}$ be the set of elements definable over $L_\alpha[U]$ by parameter-free formulas. That is, let

$$\mathfrak{M} = \{x : \text{for some formula } \varphi, L_\alpha[U] \models \varphi[x]\}.$$

We claim that $\mathfrak{M}$ is an elementary submodel of $L_\alpha[U]$. We verify that existential formulas with parameters from $\mathfrak{M}$ holding in $L_\alpha[U]$ also hold in $\mathfrak{M}$. The converse is similar. Suppose

$$L_\alpha[U] \models \exists v \varphi[v, a_1, \dots, a_n],$$

where $a_1, \dots, a_n \in \mathfrak{M}$. Since $a_1, \dots, a_n$ are definable without parameters, we can assume that φ has only one free variable. The model $L_\alpha[U]$ has a definable well-ordering $\leq_{L[U]}$ of its elements. Let ψ be the definable Skolem function for $\exists v \varphi(v)$. That is, let $\psi(v) \equiv \varphi(v) \wedge \forall v'(\varphi(v') \rightarrow v \leq_{L[U]} v')$. This formula defines an element x in $L_\alpha[U]$ without parameters, and x satisfies φ . It follows that $x \in \mathfrak{M}$ and moreover, $\mathfrak{M} \models \varphi[x]$. Thus $\mathfrak{M} \models \exists v \varphi(v)$. This establishes the claim.

Since $\mathfrak{M} \prec L_\alpha[U]$ and since the latter is the minimal model, it follows that $\mathfrak{M} = L_\alpha[U]$. Therefore, every element x of $L_\alpha[U]$ is definable in $L_\alpha[U]$ via a parameter-free formula $\varphi_x(v)$. The class of formulas $\{\varphi_x(v)\}_{x \in L_\alpha[U]}$ satisfies the theorem. ■

The property of the above class of formulas precludes getting any class of first-order indiscernibles for this model. In particular, no Prikry generic sequence can consist of first-order indiscernibles, since for each element γ of such a sequence there is a formula $\phi_{\{\gamma\}}$ satisfied only by γ .

2.4 Iterated Prikry Forcing

Magidor [5] formulated his iterated Prikry forcing to obtain results comparing compact cardinals with both measurable and supercompact cardinals. Starting with a compact cardinal κ , he forces to kill all the measurable cardinals below while

preserving the compactness of κ , thereby establishing that the first compact cardinal can be the first measurable cardinal. He also proves that the first supercompact cardinal can be the first compact cardinal.

Let Δ be a set of measurable cardinals. For each $\mu \in \Delta$, pick a normal measure U_μ on μ not containing the set of measurable cardinals below μ . Denote $\mathcal{U} = (U_\mu)_{\mu \in \Delta}$.

Definition 2.4.1. The *iterated Prikry forcing for $\mathcal{U}$* , denoted $\mathbb{M}(\mathcal{U})$, has as *conditions* pairs $\langle s_\mu, X_\mu \rangle_{\mu \in \Delta}$ such that

1. for each $\mu \in \Delta$, $s_\mu \in [\mu]^{<\omega}$
2. for each $\mu \in \Delta$, $X_\mu \in U_\mu$
3. $\bigcup_{\mu \in \Delta} s_\mu$ is finite.

A condition *extends* another, $\langle s_\mu, X_\mu \rangle_{\mu \in \Delta} \leq \langle s'_\mu, X'_\mu \rangle_{\mu \in \Delta}$, iff for each $\mu \in \Delta$ we have $\langle s_\mu, X_\mu \rangle \leq \langle s'_\mu, X'_\mu \rangle$ in the $\mathbb{P}(U_\mu)$ order.

If G is $\mathbb{M}(\mathcal{U})$ generic then for each $\mu \in \Delta$, let $\mathfrak{S}_\mu = \bigcup \{s_\mu : \exists \langle s_\mu, X_\mu \rangle_{\mu \in \Delta} \in G\}$. Each $\mathfrak{S}_\mu$ has order type ω . Magidor's forcing adds a function $\vec{\mathfrak{S}} = \{\mathfrak{S}_\mu\}_{\mu \in \Delta}$ which associates to each measurable cardinal μ in the domain a Prikry sequence $\mathfrak{S}_\mu \subset \mu$. In the work that follows we describe a forcing which adds a set of indiscernibles which are essentially the union of the range of this function, $\bigcup_{\mu \in \Delta} \mathfrak{S}_\mu$. With sufficient consistency strength some of the elements can be decoupled, so to speak, from any particular measurable cardinal.

Theorem 2.4.2. $\mathbb{M}(\mathcal{U})$ satisfies the Prikry Property.

Theorem 2.4.3. Iterated Prikry forcing preserves cardinals.

CHAPTER 3 THE FORCING

3.1 Definition

Let Δ be a set of measurable cardinals. For each $\mu \in \Delta$, pick a normal measure U_μ on μ which gives measure zero to the set of measurable cardinals below μ . Denote

$$\mathcal{U} = (U_\mu)_{\mu \in \Delta},$$

$$\kappa = \sup(\Delta).$$

These conventions will remain fixed throughout the rest of this paper. For convenience, let us assume that the only normal measures in the universe are those that appear in $\mathcal{U}$. We might as well assume that our ground model is the core model [8] for this sequence.

The following definitions are relative to $\mathcal{U}$.

Definition 3.1.1. Let the *filter of long measure one sets* be

$$\mathcal{L}(\mathcal{U}) = \{\mathcal{X} \subset \kappa : \mathcal{X} \cap \mu \in U_\mu \text{ for all } \mu \in \Delta\}.$$

Definition 3.1.2. The *disassociated indiscernible forcing*, $\mathbb{D}(\mathcal{U})$, has as *conditions* pairs $\langle s, \mathcal{X} \rangle$ such that

1. $s \in [\kappa]^{<\omega}$
2. $\mathcal{X} \in \mathcal{L}(\mathcal{U})$.

A condition *extends* another, $\langle s, \mathcal{X} \rangle \leq \langle s', \mathcal{X}' \rangle$, iff

1. $s \supset s'$,
2. $\mathcal{X} \subset \mathcal{X}'$
3. $s \setminus s' \subset \mathcal{X}'$.

For $p = \langle s, \mathcal{X} \rangle \in \mathbb{D}(\mathcal{U})$, let the *support* of p , denoted by $\text{supt}(p)$, be the set s . A condition p is a *direct extension* of q , denoted $p \leq^* q$, if $p \leq q$ and $\text{supt}(p) = \text{supt}(q)$.

The conditions and extension criteria of $\mathbb{D}(\mathcal{U})$ closely resemble those for iterated Prikry forcing (Definition 2.4.1). The main difference is that instead of getting a function associating cardinals with sets of indiscernibles, we get a *single* set of indiscernibles: denote $\mathfrak{S} = \{\eta < \kappa : \exists p \in G \text{ such that } \eta \in \text{supt}(p)\}$.

We will generally adhere to the following conventions. Lower case letters denote natural numbers $(i, j, n, \dots)$, finite sets of ordinals $(s, t, u, \dots)$, or conditions $(p, q, r, \dots)$. Upper case letters $(U, X, Z, \dots)$ denote measures or measure one sets. Upper case script letters $(\mathcal{U}, \mathcal{X}, \mathcal{Z}, \dots)$ denote sequences of measures or long measure one sets. Lower case Greek letters $(\delta, \gamma, \eta, \dots)$ usually represent ordinals.

3.2 Basic Structure

Definition 3.2.1. Relative to $\mathcal{U}$, a measure U_κ is a *vertical repeat point* if for all $X \in U_\kappa$ there exists some $\mu < \kappa$ such that $X \cap \mu \in U_\mu$.

The idea is that the measure one sets of U_κ are entangled in measure one sets from cardinals below. We will exploit this later in order to get indiscernibles that can not be associated with any particular measure.

The following proposition shows that a vertical repeat point has strictly less consistency strength than a measure of Mitchell order two.

Proposition 3.2.2. *Let $L[\mathcal{U}]$ be the minimal model for the existence of a measure of Mitchell order two on some cardinal κ . Then $\mathcal{U}(\kappa, 0)$ contains the set of cardinals $\mu < \kappa$ such that $\mathcal{U}(\mu, 0)$ is a vertical repeat point.*

Proof: The class $\mathcal{U}$ is a function such that each $\mathcal{U}(\mu, \gamma)$ is a normal measure on μ and γ is its Mitchell order index.

We first note that $\mathcal{U}(\kappa, 0)$ itself is a vertical repeat point. In fact, consider the ultrapower $j_{\mathcal{U}(\kappa, 1)} : L[\mathcal{U}] \rightarrow L[\mathcal{U}'] \approx \text{Ult}(L[\mathcal{U}], \mathcal{U}(\kappa, 1))$. Note that in $L[\mathcal{U}']$,

$j_{\mathcal{U}(\kappa,1)}(\kappa)$ is measurable. Let $X \in \mathcal{U}(\kappa, 0)$. Then $j_{\mathcal{U}(\kappa,1)}(X) \in j_{\mathcal{U}(\kappa,1)}(\mathcal{U}(\kappa, 0))$. But then $X = \kappa \cap j_{\mathcal{U}(\kappa,1)}(X) \in \mathcal{U}'(\kappa, 0)$. By elementarity, there exists some $\eta < \kappa$ such that $X \cap \eta \in \mathcal{U}(\eta, 0)$, so $\mathcal{U}(\kappa, 0)$ is a vertical repeat point. Now note that $\mathcal{U}(\kappa, 0)$ remains a vertical repeat point in $L[\mathcal{U}']$. By normality, $\kappa = [I]_{\mathcal{U}(\kappa,1)}$, where I is the identity function. The proposition then follows from Loś' Theorem. ■

Proposition 3.2.3. *Suppose $\max(\Delta) = \kappa$. That is, suppose $\sup \Delta = \kappa \in \Delta$.*

1. *If κ is a vertical repeat point then $\mathbb{D}(\mathcal{U}) = \mathbb{D}(\mathcal{U} \restriction \kappa)$.*
2. *If κ is not a vertical repeat point then forcing with $\mathbb{D}(\mathcal{U})$ is equivalent to forcing with $\mathbb{D}(\mathcal{U} \restriction \kappa) \times \mathbb{P}(U_\kappa)$, where $\mathbb{P}(U_\kappa)$ is Prikry forcing for U_κ .*

By “equivalent” in the second part above, we mean that from a generic object for one forcing, we may construct a generic object for the other.

Proof:

1. We need only show that a set that is long measure one for $\mathcal{U} \restriction \kappa$ is long measure one for $\mathcal{U}$. Let $\mathcal{X}$ be long measure one for $\mathcal{U} \restriction \kappa$. We must show that $\mathcal{X}$ has κ measure one, that is, that $\mathcal{X} \in U_\kappa$. Suppose not. Then $\kappa \setminus \mathcal{X}$ has measure one. Since U_κ is a vertical repeat point, for some $\gamma < \kappa$, we have that $\gamma \cap (\kappa \setminus \mathcal{X}) \in U_\gamma$. But then $\gamma \cap \mathcal{X} \notin U_\gamma$. This contradicts that $\mathcal{X}$ is long measure one for $\mathcal{U} \restriction \kappa$.
2. Let N be a set in U_κ witnessing that κ is not a vertical repeat point. In other words, let $N \in U_\kappa$ be such that for all $\mu < \kappa$, we have that $\mu \cap N \notin U_\mu$. Consider the map

$$i : \mathbb{D}(\mathcal{U}) \rightarrow \mathbb{D}_{\kappa \setminus N}(\mathcal{U} \restriction \kappa) \times \mathbb{P}_N^\subseteq(U_\kappa),$$

given by $\langle s, \mathcal{X} \rangle \mapsto \langle \langle s \setminus N, \mathcal{X} \setminus N \rangle, \langle s \cap N, \mathcal{X} \cap N \rangle \rangle$, where $\mathbb{D}_{\kappa \setminus N}(\mathcal{U} \restriction \kappa)$ is the disassociated indiscernible forcing for $\mathcal{U} \restriction \kappa$ below the condition $\langle 0, \kappa \setminus N \rangle$, and $\mathbb{P}_N^\subseteq(U_\kappa)$ is Prikry forcing below the condition $\langle 0, N \rangle$ with the extension of conditions, $\langle s, \mathcal{X} \rangle \leq \langle s', \mathcal{X}' \rangle$, broadened to allow $s \supset s'$ instead of just s an end

extension of s' . It is a simple matter to confirm that i is a complete embedding. That is, i preserves incompatibility and extension of conditions, and $i''\mathbb{D}(\mathcal{U})$ is a dense subset of $\mathbb{D}_{\kappa \setminus N}(\mathcal{U} \restriction \kappa) \times \mathbb{P}_N^\zeta(U_\kappa)$. It follows that the domain and target of i are equivalent as notions of forcing. The first term in the target product is obviously equivalent to $\mathbb{D}(\mathcal{U} \restriction \kappa)$, so we are left to convince the reader that $\mathbb{P}_N^\zeta(U_\kappa)$ is equivalent to Prikry forcing. Consider the map

$$\iota : \mathbb{P}(U_\kappa) \rightarrow \mathbb{P}_N^\zeta(U_\kappa)$$

given by $\langle s, X \rangle \mapsto \langle s \cap N, (X \cap N) \setminus \max(s) + 1 \rangle$. This map is also a complete embedding, and this concludes the proof of the proposition. ■

The above proposition is the first clue concerning how disassociated indiscernibles forced from vertical repeat points might behave.

3.3 Prikry Property

Definition 3.3.1. The relation $s \ll t$ means that s, t are ordinals or finite sets of ordinals (they need not be the same type of object) and that $\sup(s) < \sup(t)$.

Definition 3.3.2. Let $(\mathcal{X}_s)_{s \in [\kappa]^{<\omega}}$ be a sequence of sets. The *diagonal intersection* of this collection is the set

$$\Delta_{s \in [\kappa]^{<\omega}} \mathcal{X}_s = \{\gamma < \kappa : \forall s \ll \gamma (\gamma \in \mathcal{X}_s)\}.$$

Proposition 3.3.3. Let $\mathcal{X}_s \in \mathcal{L}(\mathcal{U})$ for $s \in [\kappa]^{<\omega}$. Then $\Delta_{s \in [\kappa]^{<\omega}} \mathcal{X}_s \in \mathcal{L}(\mathcal{U})$.

Proof: Let $\mathcal{A}$ be the diagonal intersection. Fix $\mu \in \Delta$. We will show $\mathcal{A} \cap \mu \in U_\mu$. Consider the ultrapower map $i_{U_\mu} : \mathbb{V} \rightarrow \mathfrak{M} \approx \text{Ult}(\mathbb{V}, U_\mu)$. We need to show that $\mu \in i_{U_\mu}(\mathcal{A})$. Denote $\vec{\mathcal{X}} = (\mathcal{X}_s)_{s \in [\kappa]^{<\omega}}$. It suffices to show that for $s \ll \mu$ we have $\mu \in i_{U_\mu}(\vec{\mathcal{X}})_s$. Notice that for $s \ll \mu$ we have $i_{U_\mu}(\vec{\mathcal{X}})_s = i_{U_\mu}(\mathcal{X}_s)$. Since $\mathcal{X}_s \cap \mu \in U_\mu$ for $s \ll \mu$, we have that $\mu \in i_{U_\mu}(\mathcal{X}_s)$. The theorem follows. ■

Theorem 3.3.4. *Let σ be a formula in the language of $\mathbb{D}(\mathcal{U})$ and let $\langle s, \mathcal{Z} \rangle$ be any condition. Then there exists a direct extension of $\langle s, \mathcal{Z} \rangle$ which decides σ .*

Proof: In the interest of notational simplicity, let us assume that $s = 0$. The proof can be easily modified to accommodate the more general case by carrying a constant through in some of the arguments.

Let σ and $\langle 0, \mathcal{Z} \rangle$ be given.

For each $\mu \in \Delta$ and each $s \in [\mu]^{<\omega}$, let $f_\mu^s : \mu \setminus \max(s) + 1 \rightarrow 3$ via

$$f_\mu^s(\gamma) = \begin{cases} 0 & \text{if } \exists \mathcal{X} \langle s \cup \{\gamma\}, \mathcal{X} \rangle \Vdash \sigma, \\ 1 & \text{if } \exists \mathcal{X} \langle s \cup \{\gamma\}, \mathcal{X} \rangle \Vdash \neg \sigma, \\ 2 & \text{otherwise.} \end{cases}$$

Let $H_\mu^s \in U_\mu$ be homogeneous for f_μ^s . Let $H_\mu = \Delta_{s \in [\mu]^{<\omega}} H_\mu^s$. So, for any $s \in [\mu]^{<\omega}$ and for $\gamma, \gamma' \in H_\mu \setminus \max(s) + 1$, we have that

$$\exists \mathcal{X} \langle s \cup \{\gamma\}, \mathcal{X} \rangle \parallel \sigma \text{ if and only if } \exists \mathcal{X} \langle s \cup \{\gamma'\}, \mathcal{X} \rangle \parallel \sigma,$$

and moreover, these conditions decide σ the same way.

Let $\mathcal{Z}' = \mathcal{Z} \cap \bigcup_{\mu \in \Delta} H_\mu$. We claim that a direct extension of $\langle 0, \mathcal{Z}' \rangle$ decides σ . Suppose not, and we shall reach a contradiction. Fix $\langle s \cup \{\delta\}, \mathcal{Z}'' \rangle \leq \langle 0, \mathcal{Z}' \rangle$ with $|s \cup \{\delta\}|$ the least possible such that $\langle s \cup \{\delta\}, \mathcal{Z}'' \rangle \parallel \sigma$, where $s \ll \delta$. It is possible that s is empty. Without loss of generality, assume $\langle s \cup \{\delta\}, \mathcal{Z}'' \rangle \Vdash \sigma$. Fix μ such that $\delta \in H_\mu$.

Now, for each $\gamma \in H_\mu$, there is some $\mathcal{X}$ such that $\langle s \cup \{\gamma\}, \mathcal{X} \rangle$ forces σ . Let $\mathcal{X}_\gamma$ be some such $\mathcal{X}$. Let $\mathcal{X} = \Delta_{\gamma < \mu} \mathcal{X}_\gamma$.

Let $Y = \{\nu \in H_\mu : \{\gamma < \mu : \nu \in \mathcal{X}_\gamma\} \in U_\mu\}$. Let $H = \{\gamma < \mu : \mathcal{X}_\gamma \cap \gamma = Y \cap \gamma\}$. We claim $H \in U_\mu$. If not, then let $f : \mu \setminus H \rightarrow \mu$ be such that $f(\gamma) = \nu$, where ν witnesses that $\mathcal{X}_\gamma \cap \gamma \neq Y \cap \gamma$. Then there is some $\bar{\nu}$ and $C \in U_\mu$ such that $f(\gamma) = \bar{\nu}$ for all $\gamma \in C$. Whether or not $\bar{\nu} \in Y$, this is a contradiction.

Let $\mathcal{Q} = ((Y \cup H) \cap \mathcal{X} \cap \mu) \cup (\mathcal{X} \setminus \mu)$. We claim that $\langle s, \mathcal{Q} \rangle$ forces σ , contradicting the minimality of $|s \cup \{\delta\}|$. It suffices to show that any extension of $\langle s, \mathcal{Q} \rangle$

has an extension which forces σ . Let $\langle s', Q' \rangle \leq \langle s, Q \rangle$ be arbitrary. By extending if necessary, we may assume that $s' \cap H \neq 0$. Denote $\xi = \min(s' \cap H)$. Notice that $\langle s', Q' \cap \mathcal{X}_\xi \rangle \leq \langle s \cup \{\xi\}, \mathcal{X}_\xi \rangle$. In fact, consider $\eta \in s'$ such that $\eta > \xi$. Since $\eta \in Q \subset \Delta_{\gamma < \mu} \mathcal{X}_\gamma$, we have that $\eta \in \mathcal{X}_\xi$. Now consider $\eta \in s'$ such that $\eta < \xi$. We have that $Q \cap \xi \subset (Y \cup H) \cap \xi$, and $s' \cap \xi \cap H = 0$ by choice of ξ . Since $\xi \in H$, it follows that $Y \cap \xi = \mathcal{X}_\xi \cap \xi$; so $\eta \in \mathcal{X}_\xi$ as well. Hence $s' \subset \mathcal{X}_\xi$. It follows that $\langle s', Q' \cap \mathcal{X}_\xi \rangle$ extends both $\langle s \cup \{\xi\}, \mathcal{X}_\xi \rangle$ and $\langle s', Q' \rangle$. But $\langle s \cup \{\xi\}, \mathcal{X}_\xi \rangle$ forces σ , so $\langle s', Q' \cap \mathcal{X}_\xi \rangle$ does as well. We have produced an extension of an arbitrary extension of $\langle s', Q' \rangle$ which forces σ . Hence $\langle s', Q' \rangle$ itself forces σ . This establishes the claim, which contradicts the minimality of $|s \cup \{\delta\}|$, finishing the proof. ■

3.4 More Structure

Definition 3.4.1. Let $p = \langle s, \mathcal{X} \rangle \in \mathbb{D}(\mathcal{U})$ and let $\nu < \kappa$. Let

1. $p \upharpoonright (\nu + 1) = \langle s \cap (\nu + 1), \mathcal{X} \cap (\nu + 1) \rangle$
2. $\mathbb{D}_{\nu+1}(\mathcal{U}) = \mathbb{D}(\mathcal{U} \upharpoonright (\nu + 1))$.
3. $p \downharpoonright \nu = \langle s \setminus (\nu + 1), \mathcal{X} \setminus (\nu + 1) \rangle$.
4. $\mathbb{D}^\nu(\mathcal{U}) = \{ \langle s, \mathcal{X} \rangle \in \mathbb{D}(\mathcal{U} \upharpoonright (\kappa \setminus (\nu + 1))) : (s \cup \mathcal{X}) \cap (\nu + 1) = 0 \}$.

We refer to $p \upharpoonright (\nu + 1)$ and $p \downharpoonright \nu$ as ‘ p up through $\nu + 1$ ’ and ‘ p down past ν ’, respectively.

Proposition 3.4.2. *For any $\nu < \kappa$, we have that $\mathbb{D}(\mathcal{U})$ is isomorphic to the product $\mathbb{D}_{\nu+1}(\mathcal{U}) \times \mathbb{D}^\nu(\mathcal{U})$ by the map $p \mapsto (p \upharpoonright (\nu + 1), p \downharpoonright \nu)$.*

Proposition 3.4.3. *Fix $\nu \leq \kappa$.*

1. $\mathbb{D}_{\nu+1}(\mathcal{U})$ has the μ^+ -chain condition, where $\mu = |\sup(\Delta \cap (\nu + 1))|$.
2. The $\leq^*$ relation is 2^ν -closed in $\mathbb{D}^\nu(\mathcal{U})$.

Proof: For the first assertion, notice that incompatible elements must differ in their support. Since there are only μ choices for the support of a condition in $\mathbb{D}_{\nu+1}(\mathcal{U})$, there can be at most μ incompatible conditions.

For the second assertion, note that each member of $\Delta \setminus (\nu + 1)$ is greater than ν and a strong limit. Fix $\eta < \lambda$. The claim follows from the fact that if $\mathcal{X}_\gamma \in \mathcal{L}(\mathcal{U} \upharpoonright (\kappa \setminus (\nu + 1)))$ for $\gamma < \eta$, then $\cap_{\gamma < \eta} \mathcal{X}_\gamma \in \mathcal{L}(\mathcal{U} \upharpoonright (\kappa \setminus (\nu + 1)))$. ■

Theorem 3.4.4. *Fix $\nu < \kappa$. Let σ be a formula in the language of $\mathbb{D}_{\nu+1}(\mathcal{U}) \times \mathbb{D}^\nu(\mathcal{U})$. Let $(p, q) \in \mathbb{D}_{\nu+1}(\mathcal{U}) \times \mathbb{D}^\nu(\mathcal{U})$. There is a $q' \leq^* q$ such that for any $(p', q'') \leq (p, q')$ such that (p', q'') decides σ , we have that (p', q') decides σ (the same way that (p', q'') does.)*

Proof: For each $a \in \mathbb{D}_{\nu+1}(\mathcal{U})$, there is a $q(a) \leq^* q$, $q(a) \in \mathbb{D}^\nu(\mathcal{U})$ such that $(a, q(a))$ decides the formulas

$$\exists q \in \dot{G}^\nu (a, q) \Vdash \sigma \quad (\mathbf{T}_a)$$

$$\exists q \in \dot{G}^\nu (a, q) \Vdash \neg \sigma. \quad (\mathbf{F}_a)$$

Since, in $\mathbb{D}^\nu(\mathcal{U})$, the direct extension relation is 2^ν -closed, there is some q' such that $q' \leq q(a)$ for all $a \in \mathbb{D}_{\nu+1}(\mathcal{U})$. We claim that this q' satisfies the theorem. Suppose $(p', q'') \leq (p, q')$ such that (p', q'') decides σ . Without loss of generality, suppose $(p', q'') \Vdash \sigma$. It follows that (p', q'') forces $\mathbf{T}_{p'}$. Now consider (p', q') . Since $(p', q') \leq (p, q(p))$, we have that (p', q') decides $\mathbf{T}_{p'}$. But (p', q') is compatible with (p', q'') . We have that (p', q') decides $\mathbf{T}_{p'}$ and is compatible with a condition that forces $\mathbf{T}_{p'}$. Hence (p', q') must also force $\mathbf{T}_{p'}$. With this fact in hand, let us suppose for a contradiction that (p', q') does not force σ . Since (p', q') does not force σ , there is a $(p'', q'') \leq (p', q')$ such that $(p'', q'') \Vdash \neg \sigma$. Since (p'', q'') extends (p', q') , we have that (p'', q'') also forces $\mathbf{T}_{p'}$. This means that for some q''' compatible with q'' , the condition (p, q''') forces σ .

But (p'', q'') is compatible with (p, q''') , and the former forces $\neg\sigma$ while the latter forces σ . This contradiction establishes the theorem. ■

Theorem 3.4.5. *Disassociated indiscernible forcing preserves cardinals.*

Proof: Suppose for a contradiction that λ is the least cardinal in $\mathbf{V}$ that is not a cardinal in $\mathbf{V}[G]$.

First note that λ is a successor cardinal since if all cardinals below a limit cardinal are preserved then the limit cardinal must be preserved as well. Hence the least collapsed cardinal cannot be a limit.

Next, $\lambda < \kappa$. In fact, $\mathbb{D}(\mathcal{U})$ has the κ^+ chain condition and hence preserves cardinals above κ .

We now claim that λ is not a cardinal in $\mathbf{V}[G](\lambda + 1)$. To see this, we demonstrate that the power set of λ is the same in $\mathbf{V}[G]$ and $\mathbf{V}[G](\lambda + 1)$. Let $\dot{S}$ be a $\mathbb{D}(\mathcal{U})$ -name for a subset of λ . Suppose $p \Vdash \dot{S} \subset \lambda$. For each $\nu < \lambda$, apply Theorem 3.4.4 to p at $\lambda + 1$ for the formula $\ulcorner \nu \in \dot{S} \urcorner$ and get a condition $q(\nu) \in \mathbb{D}_{\lambda+1}(\mathcal{U})$. Since the $q(\nu)$ have identical support, by Proposition 3.4.3 there is a condition q extending all of them. Consider $\dot{T}$, the name for the set $\{\nu < \lambda : (p \restriction (\lambda + 1), q) \Vdash \nu \in \dot{S}\}$. It follows from the definition that $\dot{S}^G = \dot{T}^{G(\lambda+1)}$, which establishes the claim.

But now notice that $\mathbb{D}_{\lambda+1}(\mathcal{U})$ has the λ chain condition. Hence λ must remain a cardinal in $\mathbf{V}[G](\lambda + 1)$, which is a contradiction. ■

3.5 Genericity Criteria

Theorem 3.5.1. (Genericity Criteria) *Let $\mathfrak{S} \subset \kappa$. Then $\mathfrak{S}$ is $\mathbb{D}(\mathcal{U})$ generic if and only if for all $\mathcal{X} \subset \kappa$,*

$$|\mathfrak{S} \setminus \mathcal{X}| < \aleph_0 \iff \mathcal{X} \in \mathcal{L}(\mathcal{U}). \quad (3.1)$$

Proof: The proof of this theorem will take up the whole section. Clearly, if G is $\mathbb{D}(\mathcal{U})$ generic, then $\mathfrak{S} = \cup\{s : \langle s, \mathcal{X} \rangle \in G\}$ satisfies (3.1). Suppose now that

$\mathfrak{S}$ satisfies (3.1). Let $G[\mathfrak{S}] = \{\langle s, \mathcal{X} \rangle : s \in \mathfrak{S} \text{ and } \mathfrak{S} \setminus s \subset \mathcal{X}\}$. Let $\mathcal{D}$ be dense and open in $\mathbb{D}(\mathcal{U})$. We shall prove that $\mathcal{D} \cap G[\mathfrak{S}] \neq \emptyset$.

Definition 3.5.2. For any dense D , let

$$D^* = \{\langle s, \mathcal{X} \rangle : \exists \mu \in \Delta \setminus \max(s) \{ \gamma < \mu : \langle s \cup \{\gamma\}, \mathcal{X} \rangle \in D \} \in U_\mu\}.$$

The following is the main technical lemma used in the proof.

Lemma 3.5.3. *Let D be dense open. There is a long measure one set $Q(D)$ such that for any t , if δ is the least possible ordinal such that $t \ll \delta$ and $\langle 0, Q(D) \rangle \geq \langle t \cup \{\delta\}, \mathcal{X} \rangle \in D$ for some $\mathcal{X}$, then $\langle t, Q(D) \rangle \in D^*$.*

Proof: The issue here is that the long measure one part $Q(D)$ of the condition in the conclusion is fixed. We obtain this set by exploiting the fact that D is open; a large part of the proof consists of getting $Q(D)$ contained in enough sets $\mathcal{X}$ such that $\langle t, \mathcal{X} \rangle \in D$. The set D^* provides the relevant concept of ‘enough.’ As in the proof of the Prikry Property, we begin by labeling the relevant long measure one sets. For each $t \in [\kappa]^{<\omega}$, if there exists an $\mathcal{X}$ such that $\langle t, \mathcal{X} \rangle \in D$ then let $\mathcal{X}_t$ be some such $\mathcal{X}$. Since D is open, the goal becomes finding a set $Q(D)$ that is contained in enough sets $\mathcal{X}_t$.

Let us first take care of getting a set $\mathcal{E}$ eventually contained in each $\mathcal{X}_t$. Let

$$\mathcal{E} = \Delta_{t \ll \kappa} \mathcal{X}_t.$$

Then for each $t \in [\kappa]^{<\omega}$ we have

$$\mathcal{E} \setminus \max(t) + 1 \subset \mathcal{X}_t. \tag{3.2}$$

We will use this to ensure that enough sets $\mathcal{X}_t$ contain the part of $Q(D)$ above t . Getting enough sets $\mathcal{X}_t$ containing the part of $Q(D)$ below t will prove more difficult.

We continue by defining measure one sets H each of which is homogeneous for getting a condition in D . For each $\mu \in \Delta$, each $t \in [\kappa]^{<\omega}$ such that $t \ll \mu$, let $f_\mu^t : \mu \rightarrow 2$ such that

$$f_\mu^t(\gamma) = \begin{cases} 0 & \text{if } \exists \mathcal{X} \langle t \cup \{\gamma\}, \mathcal{X} \rangle \in D, \\ 1 & \text{otherwise.} \end{cases}$$

Let $H_\mu^t \in U_\mu$ be homogeneous for f_μ^t and maximal. Let

$$H_\mu = \Delta_{t \in [\mu]^{<\omega}} H_\mu^t.$$

Thus, if $\gamma, \gamma' \in H_\mu$ and $t \ll \gamma, \gamma'$ then

$$\exists \mathcal{X} \langle t \cup \{\gamma\}, \mathcal{X} \rangle \in D \iff \exists \mathcal{X} \langle t \cup \{\gamma'\}, \mathcal{X} \rangle \in D. \quad (3.3)$$

We can now define sets $\mathcal{J}$ and W with the property that, roughly speaking, W is homogeneous for indexing long measure one sets $\mathcal{X}$ that contain an initial part of $\mathcal{J}$ - see (3.4).

First we need some auxiliary notation. Define a partial function $\mu : [\kappa]^{<\omega} \rightarrow \Delta$ as follows. Let $\mu(t)$ be the least μ such that $\{\gamma : \exists \mathcal{X} \langle t \cup \{\gamma\}, \mathcal{X} \rangle \in D\} \in U_\mu$. All of the following presumes that this function is defined. Note that $\mu(t)$ is the least μ (if it exists) such that $f_\mu^{t''} H_\mu^t = \{0\}$.

For each $t \in [\kappa]^{<\omega}$ and each $\eta < \mu(t)$, let $g_\eta^t : \mu(t) \rightarrow \mathcal{P}(\eta)$ be such that

$$g_\eta^t(\gamma) = \mathcal{X}_{t \cup \{\gamma\}} \cap \eta.$$

Let $W_\eta^t \in U_{\mu(t)}$ be homogeneous for g_η^t . Let

$$W^t = \Delta_{\eta < \mu(t)} W_\eta^t.$$

Let

$$J^t = \bigcup_{\gamma \in W^t} (\gamma \cap \mathcal{X}_{t \cup \{\gamma\}}).$$

Each J^t is long measure one up to $\mu(t)$. To get a long measure one set up to κ , let $\mathcal{J}^t$ be J^t with the interval $[\mu(t), \kappa)$ adjoined. Let

$$\mathcal{J} = \Delta_{t \ll \kappa} \mathcal{J}^t.$$

We have, for any t such that W^t is defined and for $\gamma \in W^t$,

$$\mathcal{X}_{t \cup \{\gamma\}} \supset \mathcal{J} \cap \gamma. \quad (3.4)$$

This will ensure that each $\mathcal{X}_{t \cup \{\gamma\}}$ contains the part of $\mathcal{Q}(D)$ below γ .

Finally, let

$$\mathcal{Q}(D) = \mathcal{E} \cap \mathcal{J} \cap \bigcup_{\mu \in \Delta} H_\mu.$$

We claim that $\mathcal{Q}(D)$ satisfies the lemma. For some t , δ , and $\mathcal{X}$, we have that $\langle 0, \mathcal{Q}(D) \rangle \geq \langle t \cup \{\delta\}, \mathcal{X} \rangle \in D$; pick t and δ so that $t \ll \delta$ and so that δ is as small as possible. We claim that this condition is in D^* . In fact, we show

$$\{\gamma : \langle t \cup \{\gamma\}, \mathcal{Q}(D) \rangle \in D\} \in U_{\mu(t)}.$$

Note that $\delta \in H_\mu^t$ for some μ with $f_\mu^{t^n} H_\mu^t = \{0\}$. By minimality of δ , we have that $\mu = \mu(t)$. Furthermore, (3.3) implies that

$$\{\gamma : \langle t \cup \{\gamma\}, \mathcal{X}_{t \cup \{\gamma\}} \rangle \in D\} \in U_{\mu(t)}.$$

By (3.2) and (3.4) we have that $\mathcal{Q}(D)$ is contained in each $\mathcal{X}_{t \cup \{\gamma\}}$ above. Since D is open, we have that $\langle t \cup \{\gamma\}, \mathcal{Q}(D) \rangle \in D$ for γ in a measure one set. This concludes the proof of the lemma. ■

Let us define the dense sets to which we will apply Lemma 3.5.3.

Definition 3.5.4. For each $s \in [\kappa]^{<\omega}$, recursively define $D_{s,n}$ for $n \in \omega$ via

$$D_{s,0} = \{\langle s', \mathcal{X} \rangle : \langle s \cup s', \mathcal{X} \rangle \in \mathcal{D}\},$$

$$D_{s,n+1} = D_{s,n}^*.$$

We will use these sets to get a condition that is in $G[\mathfrak{S}] \cap D_{0,m}$ for some $m \in \omega$. Let $s_0 = \mathfrak{S} \setminus \mathcal{Q}(D_{0,0})$. Then $\langle s_0, \mathcal{Q}(D_{0,0}) \rangle \in G[\mathfrak{S}]$. Suppose now that $\bar{s}_i = (s_0, \dots, s_i)$ has been defined. For notational simplicity, let us identify $\bar{s}_i$ with the set $s_0 \cup \dots \cup s_i$. Let

$$\mathcal{Z}(\bar{s}_i) = \cap_{j \leq i} \cap_{n < \omega} \Delta_{s \ll \kappa} \mathcal{Q}(D_{\bar{s}_j \cup s, n}),$$

$$s_{i+1} = \mathfrak{S} \setminus (\mathcal{Z}(\bar{s}_i) \cup \bar{s}_i).$$

Then $\langle \bar{s}_{i+1}, \mathcal{Z}(\bar{s}_i) \rangle \in G[\mathfrak{S}]$, where $\bar{s}_{i+1} = (s_0, \dots, s_{i+1})$.

Claim 1: For each $i < \omega$ we have $s_{i+1} \ll s_i$.

Proof: Denote $\delta = \max(s_{i+1})$. By definition of $\bar{s}_i$, we have that $\delta \notin \mathcal{Z}(\bar{s}_i)$. However, since $\mathcal{Z}(\bar{s}_i)$ is a diagonal intersection, if $\max(s_i) < \delta$, then $\delta \in \mathcal{Z}(\bar{s}_i)$. This impossibility establishes the claim. ■

Since our universe is well founded, infinite descending chains of membership are excluded. Hence for some $k < \omega$ we have that $s_{k+1} = 0$.

Claim 2: For some $m < \omega$, we have $\langle \bar{s}_{k+1}, \mathcal{Z}(\bar{s}_k) \rangle \in D_{0,m}$.

Proof: For some $\{\delta_1, \dots, \delta_m\}$ and some $\mathcal{X}$, we have $\langle 0, \mathcal{Z}(\bar{s}_k) \rangle \geq \langle \{\delta_1, \dots, \delta_m\}, \mathcal{X} \rangle \in D_{\bar{s}_{k+1}, 0}$. We show by induction on $i \leq m$ that $\langle \{\delta_1, \dots, \delta_{m-i}\}, \mathcal{Z}(\bar{s}_k) \rangle \in D_{\bar{s}_{k+1}, i}$. Suppose that $\langle \{\delta_1, \dots, \delta_{m-i}\}, \mathcal{Z}(\bar{s}_k) \rangle \in D_{\bar{s}_{k+1}, i}$. Pick δ'_{m-i} the least possible such that for some $\mathcal{X}$

$$\langle 0, \mathcal{Z}(\bar{s}_k) \rangle \geq \langle \{\delta_1, \dots, \delta_{m-i-1}, \delta'_{m-i}\}, \mathcal{X} \rangle \in D_{\bar{s}_{k+1}, i}.$$

Since $\mathcal{Q}(D_{\bar{s}_{k+1}, i}) \subset \mathcal{Z}(\bar{s}_k)$, we may apply Lemma 3.5.3 and use the fact that $D_{\bar{s}_{k+1}, i}$ is open to see that

$$\langle \{\delta_1, \dots, \delta_{m-(i+1)}\}, \mathcal{Z}(\bar{s}_k) \rangle \in D_{\bar{s}_{k+1}, i+1}.$$

Hence

$$\langle \bar{s}_{k+1} \cup \{\delta_1, \dots, \delta_{m-i-1}\}, \mathcal{Z}(\bar{s}_k) \rangle \in D_{0, i+1}.$$

This establishes the claim ■

Finally, we claim that for some $\{\gamma_1, \dots, \gamma_m\}$ we have that

$$\langle \bar{s}_{k+1} \cup \{\gamma_1, \dots, \gamma_m\}, \mathcal{Z}(\bar{s}_k) \rangle \in D_{0,0} \cap G[\mathfrak{S}].$$

We proceed by induction on $i \leq m$. Suppose

$$\langle \bar{s}_{k+1} \cup \{\gamma_1, \dots, \gamma_i\}, \mathcal{Z}(\bar{s}_k) \rangle \in D_{0,m-i} \cap G[\mathfrak{S}].$$

Select γ_{i+1} as follows. Since $D_{0,m-i} = D_{0,m-i-1}^*$, there is a measure one set of elements γ such that adding γ to the support of the above condition gets the condition into $D_{0,m-i-1}$. Since $\mathfrak{S}$ is unbounded in U_μ for all $\mu \in \Delta$ by hypothesis, we may select γ_{i+1} from the aforementioned measure one set intersected with $\mathfrak{S}$. The augmented condition will be in $D_{0,m-i-1} \cap G[\mathfrak{S}]$. This completes the induction and establishes the claim.

We have found a condition in both $\mathcal{D}$ and $G[\mathfrak{S}]$, and since $\mathcal{D}$ was an arbitrary dense open set it follows that $G[\mathfrak{S}]$ is generic. This concludes the proof of Theorem 3.5.1. ■

CHAPTER 4 GETTING TWO NORMAL MEASURES

4.1 The Measures

Assume that U_κ is a vertical repeat point. We show that after disassociated indiscernible forcing not only does κ remain measurable, but in fact κ has exactly two normal measures. In this section we really do need to assume that our ground model is the minimal model having a repeat point.

Definition 4.1.1. We define $\dot{U}_\kappa^i$ for $i \in \{0, 1\}$. Let $j : \mathbf{V} \rightarrow \mathfrak{M} = \text{Ult}(\mathbf{V}, U_\kappa)$ be the ultrapower map via the normal measure U_κ . A condition $\langle s, \mathcal{X} \rangle$ in the generic filter forces a set $\dot{X}$ to be in $\dot{U}_\kappa^0$, (ie. $\langle s, \mathcal{X} \rangle \Vdash \dot{X} \in \dot{U}_\kappa^0$) iff for some $\mathcal{X}' \in \mathcal{L}(j(\mathcal{U}))$ with $\mathcal{X}' \cap \kappa = \mathcal{X}$ we have $\langle s, \mathcal{X}' \rangle \Vdash \kappa \in j(\dot{X})$. Similarly, $\langle s, \mathcal{X} \rangle$ forces $\dot{X}$ to be in $\dot{U}_\kappa^1$ iff for some $\mathcal{X}' \in \mathcal{L}(j(\mathcal{U}))$ with $\mathcal{X}' \cap \kappa = \mathcal{X}$ we have $\langle s \cup \{\kappa\}, \mathcal{X}' \rangle \Vdash \kappa \in j(\dot{X})$. In either case, the condition which forces κ to be in $\dot{X}$ is said to be a *witness of $\langle s, \mathcal{X} \rangle$ forcing X in U_κ^i* .

Here is an informal discussion of why the definition works and why a vertical repeat point is required. Note that a condition q which witnesses that p forces X to be in U_κ^i is compatible with $j(p)$. In fact we require the long measure one part of p to match the long measure one part of q up to κ . Since U_κ is a vertical repeat point, long measure one sets in ultrapowers by any $U_\kappa^* \supset U_\kappa$ are measure one for U_κ (even though κ cannot be measurable in any such ultrapower). Hence the long measure one part of q is measure one for U_κ . This is key. Since U_κ is a vertical repeat point, we have that $q \restriction (\kappa + 1)$ is in the original forcing for $\mathcal{U}$. In other words, the fact that U_κ is a vertical repeat point ensures that the restriction to $\kappa + 1$ of images of conditions

in $\mathbb{D}(\mathcal{U})$ remain in $\mathbb{D}(\mathcal{U})$. Further, if U_κ were not a vertical repeat point we could have some q (compatible with the image of p) forcing the part of the set of indiscernibles below κ to be disjoint from a set in U_κ . But the set of indiscernibles must be identical to its image up to κ under any ultrapower with critical point κ . In other words, for a condition p to force the set of indiscernibles to be in some normal measure extending U_κ , conditions compatible with the image of p must have a long measure one part which ensures that the initial segment of the indiscernibles comes from sets in U_κ . Otherwise, the part of the image of the indiscernibles below the critical point could differ from the original set of indiscernibles.

Theorem 4.1.2. *In the generic extension, U_κ^0 and U_κ^1 are normal κ -complete non-principle ultrafilters with $\mathfrak{S} \notin U_\kappa^0$ and $\mathfrak{S} \in U_\kappa^1$.*

Proof: We first need to show that the U_κ^i are well-defined. That is, $p \Vdash \dot{X} \in U_\kappa^i$ does not depend on choice of name for X . Suppose $p \Vdash \dot{X} = \dot{Y}$ and $p \Vdash \dot{X} \in U_\kappa^i$. Then $j(p) \Vdash j(\dot{X}) = j(\dot{Y})$, and if $q \leq j(p)$ with $q \Vdash \kappa \in \dot{X}$, then $q \Vdash \kappa \in \dot{Y}$. So the definition is name-independent.

Next, each U_κ^i is a nonprinciple ultrafilter. We show that U_κ^i is closed under enlargements; the proofs of closure under intersection, nonprincipality, and maximality are similar. Suppose $p \Vdash \ulcorner X \in U_\kappa^i \wedge X \subset Y \urcorner$. There is a witness $q \leq j(p)$ of p forcing X in U_κ^i . So $q \Vdash \ulcorner \kappa \in j(X) \wedge j(X) \subset j(Y) \urcorner$. It follows that $q \Vdash \kappa \in j(Y)$. Hence q is also a witness of p forcing Y in U_κ^i .

Let us now show that U_κ^i is κ -complete. Suppose for some $\eta < \kappa$ we have $p \Vdash \bigcup_{\gamma < \eta} X_\gamma = \kappa$. We find a $p' \leq p$ and a $\gamma' < \eta$ such that $p' \Vdash X_{\gamma'} \in U_\kappa^i$. By elementarity, $j(p) \Vdash \bigcup_{\gamma < \eta} j(X_\gamma) = j(\kappa)$. For each $\gamma < \eta$, apply Theorem 3.4.4 to $j(p)$ at κ to get a pair $(r_\kappa, r^\kappa(\gamma)) \in \mathbb{D}_{\kappa+1}(j(\mathcal{U})) \times \mathbb{D}^\kappa(j(\mathcal{U}))$ such that $r_\kappa = j(p) \restriction (\kappa + 1)$, $r^\kappa(\gamma) \leq^* j(p) \restriction \kappa$, and such that if $(a, b) \leq (r_\kappa, r^\kappa(\gamma))$ with (a, b) deciding $\kappa \in j(X_\gamma)$, then $(a, r^\kappa(\gamma))$ decides $\kappa \in j(X_{\gamma'})$ the same way. Since $\mathbb{D}^\kappa(j(\mathcal{U}))$ is κ^+ -closed and

the $r^\kappa(\gamma)$ are compatible, there is a condition $r^\kappa \leq r^\kappa(\gamma)$ for all $\gamma < \eta$. Note that $r_\kappa = j(p) \restriction (\kappa + 1)$ for all $\gamma < \eta$, so that $(j(p) \restriction (\kappa + 1), r^\kappa) \leq (r_\kappa, r^\kappa(\gamma))$ for all $\gamma \leq \eta$. Now since $(r_\kappa, r^\kappa) \Vdash \bigcup_{\gamma < \eta} j(X_\gamma) = j(\kappa)$, for some $q \leq (p', r^\kappa)$ and some $\gamma' < \eta$, we have that $q \Vdash \kappa \in j(X_{\gamma'})$. Denote $p' = q \restriction (\kappa + 1)$. Since U_κ is a vertical repeat point, $p' \in \mathbb{D}(\mathcal{U})$. We claim that (p', r^κ) is a witness to p' forcing $X_{\gamma'}$ in U_κ^i , where the i depends on whether or not κ is in the support of q . It is clear from the fact that $r^\kappa \leq^* j(p) \restriction \kappa$ that the support of (p', r^κ) is equal to the support of p' . Also $j((p', r^\kappa)) \restriction (\kappa + 1) = j(p') \restriction (\kappa + 1)$. And by definition of the (r_κ, r^κ) we have that $(p', r^\kappa) \Vdash \kappa \in (X_{\gamma'})$. Thus (p', r^κ) is a witness as claimed. It follows that $p' \Vdash X_{\gamma'} \in U_\kappa^i$ so that U_κ^i is κ -complete.

Finally, we show that the U_κ^i are normal. Suppose $p \Vdash \ulcorner f : \kappa \rightarrow \kappa \wedge \forall \gamma < \kappa \ f(\gamma) < \gamma \urcorner$. We find a $p' \leq p$, a $\gamma' < \kappa$, and a set X such that $p' \Vdash \ulcorner X \in U_\kappa^i \wedge f''X = \gamma' \urcorner$. For every $\gamma < \kappa$, apply Theorem 3.4.4 to $j(p)$ at κ to get a pair $(r_\kappa, r^\kappa(\gamma)) \in \mathbb{D}_{\kappa+1}(j(\mathcal{U})) \times \mathbb{D}^\kappa(j(\mathcal{U}))$ such that $r_\kappa = j(p) \restriction (\kappa + 1)$, $r^\kappa(\gamma) \leq^* j(p) \restriction \kappa$, and such that if $(a, b) \leq (r_\kappa, r^\kappa(\gamma))$ with (a, b) deciding $j(f)(\kappa) = \gamma$, then $(a, r^\kappa(\gamma))$ decides $j(f)(\kappa) = \gamma$ the same way. Since $\mathbb{D}^\kappa(j(\mathcal{U}))$ is κ -closed, there is a condition $r^\kappa \leq r^\kappa(\gamma)$ for all $\gamma < \eta$. Now, since $j(p) \geq (r_\kappa, r^\kappa) \Vdash j(f) : j(\kappa) \rightarrow j(\kappa)$, there is some $\gamma' < \kappa$ and some $q \leq (r_\kappa, r^\kappa)$ such that $q \Vdash j(f)(\kappa) = \gamma'$. Denote $p' = q \restriction (\kappa + 1)$, which is in $\mathbb{D}(\mathcal{U})$ since U_κ is a vertical repeat point. We claim that (p', r^κ) is a witness to p' forcing $\{\gamma < \kappa : f(\gamma) = \gamma'\}$ in U_κ^i . It is clear from the fact that $r^\kappa \leq^* j(p) \restriction \kappa$ that the support of (p', r^κ) is equal to the support of p' . Also $j((p', r^\kappa)) \restriction (\kappa + 1) = j(p') \restriction (\kappa + 1)$. And by definition of the (r_κ, r^κ) we have that $(p', r^\kappa) \Vdash j(f)(\kappa) = \gamma'$. In other words, $(p', r^\kappa) \Vdash \kappa \in \{\gamma < j(\kappa) : j(f)(\gamma) = \gamma'\}$. That (p', r^κ) is the witness as claimed follows by elementarity. ■

Theorem 4.1.3. *The only normal ultrafilters on κ in the generic extension are U_κ^0 and U_κ^1 .*

Proof: Suppose $p = \langle s, \mathcal{X} \rangle \Vdash \ulcorner X \in W \urcorner \wedge W$ is a normal measure on $\kappa^\intercal$. We find a $q \leq p$ such that $q \Vdash X \in U_\kappa^i$ for some $i \in \{0, 1\}$.

In the generic extension there is an ultrapower map j_W . The restriction of this function to the ground model factors through the ultrapower $j = j_{U_\kappa}$.

We appeal to Theorem 3.4.4. For each $\nu < \kappa$ there is a $q_\nu = \langle s, \mathcal{Q}_\nu \rangle \leq^* p \restriction \nu$ in $\mathbb{D}^\nu(\mathcal{U})$ such that if $(a, b) \leq (1, q_\nu)$ and $(a, b) \in \mathbb{D}_{\nu+1}(\mathcal{U}) \times \mathbb{D}^\nu(\mathcal{U})$ decides $\nu \in X$ then (a, q_ν) decides $\nu \in X$ the same way.

Let $\mathcal{Q} = \Delta_{\nu < \kappa}$ and let $p' = \langle s, \mathcal{X} \cap \mathcal{Q} \rangle$.

Now $p' \Vdash X \in W$. Hence for any generic filter G with $p' \in G$ we have $V[G'] \models \kappa \in j(X)$, where $j(p') \in G'$. It follows that there is some q compatible with $j(p)$ such that $q \Vdash \kappa \in j(X)$. Denote $q' = (q \restriction (\kappa + 1), j(p') \restriction \kappa)$. Note that q' is also compatible with $j(p')$. We can assume that $q' \restriction \kappa \leq p'$.

Then $q' \Vdash X \in U_\kappa^i$, where

$$i = \begin{cases} 0 & \text{if } \kappa \in \text{supt}(q \restriction (\kappa + 1)), \\ 1 & \text{if } \kappa \notin \text{supt}(q \restriction (\kappa + 1)). \end{cases}$$

By the elementarity of j and the definition of p' , we have $q' \Vdash \kappa \in j(X)$. The claim, and hence the theorem, follows. ■

4.2 Properties

Theorem 4.2.1. *In the generic extension, the following hold:*

1. *For each $X \in U_\kappa^0$ there is some $Y \in U_\kappa$ such that $Y \setminus \mathfrak{S} \subset X$.*
2. *For each $X \in U_\kappa^1$ there is some $Y \in U_\kappa$ such that $Y \cap \mathfrak{S} \subset X$.*

Proof: Suppose $p = \langle s, \mathcal{X} \rangle \Vdash X \in U_\kappa^i$ for some $i \in \{0, 1\}$. The two cases are not exclusive, but without loss of generality we can assume that exactly one holds. By Theorem 3.4.4, for each $\nu < \kappa$, there are $q_\nu = \langle s \setminus (\nu + 1), \mathcal{Q}_\nu \rangle \in \mathbb{D}^\nu(\mathcal{U})$ such that if $(a, q') \leq (1, q_\nu)$, where $(a, q') \in \mathbb{D}_{\nu+1}(\mathcal{U}) \times \mathbb{D}^\nu(\mathcal{U})$ such that (a, q') decides $\nu \in X$,

then (a, q_ν) decides $\nu \in X$ the same way that (a, q') does. In other words, q_ν decides $\nu \in X$ above ν . Let $\mathcal{Q} = \Delta_{\nu < \kappa} \mathcal{Q}_\nu$ and let $q = \langle s, \mathcal{X} \cap \mathcal{Q} \rangle$.

Since $q \Vdash X \in U_\kappa^i$, there is an $r \leq j(q)$ such that $r \restriction (\kappa + 1) = j(q) \restriction (\kappa + 1)$, $r \Vdash \kappa \in j(X)$, and $\text{supt}(r) \setminus \text{supt}(q)$ is either $\{0\}$ or $\{\kappa\}$, depending on whether X is forced to be in U_κ^0 or U_κ^1 by p . In the former case, let $r_\nu^0 \in \mathbb{D}_\nu(\mathcal{U})$ be the restriction of r to $\nu + 1$. In the latter case, let r_ν^1 be the above restriction augmented by adding ν to its support.

Let $Y_i = \{\nu < \kappa : (r_\nu^i, q \restriction \nu) \Vdash \nu \in X\}$. Note that $j(Y_i) = \{\nu < j(\kappa) : (j(r)_\nu^i, j(q) \restriction \nu) \Vdash \nu \in j(X)\}$. Since $j(r)_\kappa^i = r \restriction (\kappa + 1)$ and $(r \restriction (\kappa + 1), j(q) \restriction \kappa) \Vdash \kappa \in j(X)$, we have that $\kappa \in j(Y)$. It follows that $Y \in U_\kappa$.

Suppose now $q' \leq q$ and either $q' \Vdash \nu \in Y \cap \mathfrak{S}$ or $q' \Vdash \nu \in Y \setminus \mathfrak{S}$. Note that the above decides whether or not $\nu \in \text{supt}(q')$. In either case, we have $(r_\nu^i, q \restriction \nu) \Vdash \nu \in X$ by definition of Y . Since $r_\nu^i = q \restriction (\nu + 1) \geq q' \restriction (\nu + 1)$, we have $(q' \restriction (\nu + 1), q \restriction \nu) \Vdash \nu \in X$. Since $q' \restriction \nu \leq q \restriction \nu$, we have $q' \Vdash \nu \in X$. The theorem follows. ■

Compare the following with Theorem 2.3.6.

Theorem 4.2.2. *For $i \in \{0, 1\}$, consider the restriction of the ultrapower*

$$j_{U_\kappa^i} : L[\mathcal{U}][\mathfrak{S}] \rightarrow L[\mathcal{U}'][\mathfrak{S}'] \approx \text{Ult}(L[\mathcal{U}][\mathfrak{S}], U_\kappa^i)$$

to the ground model

$$j_i : L[\mathcal{U}] \rightarrow L[\mathcal{U}'].$$

Then

1. j_i is an iterated ultrapower along measures from $\mathcal{U}$
2. every element of $\mathfrak{S}' \setminus \mathfrak{S}$ is a critical point of the iterated ultrapower.

Proof: The first point is established in [8]. Let us demonstrate the second. Suppose for a contradiction that $\nu \in \mathfrak{S}' \setminus \mathfrak{S}$ is not a critical point in the iteration. Then for some function $f : \kappa \rightarrow \kappa$, there is a $\gamma < \nu$ such that $\nu = j_{U_\kappa^i}(f)(\gamma)$. Thus ν is in the set $\{\eta < j_{U_\kappa^i}(\kappa) : \eta \in j_{U_\kappa^i}(f)''\eta\}$. In other words $\nu \in j_{U_\kappa^i}(Z)$, where $Z = \{\eta < \kappa : \eta \in f''\eta\}$. Note that for all $\mu \in \Delta$ we have $Z \cap \mu \notin U_\mu$. Denote $D = \{\langle s, X \cap Z \rangle : \langle s, X \rangle \in \mathbb{D}(\mathcal{U})\}$. Now D is dense, so $G \cap D \neq \emptyset$ for all generic G . Suppose p is a condition in the intersection. We have $j_{U_\kappa^i}(p) \in j_{U_\kappa^i}(G)$. But since p has finite support bounded below κ , we have that ν is not in the support of $j_{U_\kappa^i}(p)$. Neither is ν in the long measure one set. Since $\mathfrak{S}' = j_{U_\kappa^i}(\mathfrak{S})$, we cannot have $\nu \in \mathfrak{S}'$. This contradiction establishes the theorem. ■

CHAPTER 5 ITERATED DISASSOCIATED INDISCERNIBLES

5.1 Definition

Definition 5.1.1. We define functions $f_\gamma : \kappa \rightarrow \kappa$ for $\gamma < \kappa^+$ by recursion. Let $f_0(\eta) = 0$. Let $f_{\gamma+1}(\eta) = f_\gamma(\eta) + 1$. For limit ordinals γ , let $f_\gamma(\eta) = \bigcup_{\nu < \gamma} f_\nu(\eta)$, where $(\gamma_\nu)_{\nu < \kappa}$ is onto γ .

The definition of f_γ for limit γ is the diagonal union. Note that $[f_\gamma]_{U_\kappa} = \gamma$. That is, these functions are canonical representations of the ordinals less than κ^+ in the ultrapower by U_κ .

The following is relative to our fixed sequence of measures $\mathcal{U} = (U_\mu)_{\mu \in \Delta}$.

Definition 5.1.2. We define *vertical repeat point of order $\gamma \leq \kappa^+$* by recursion on γ . Let $\text{ord}(U_\kappa) = 0$ if for some $X \in U_\kappa$, for all $\mu < \kappa$ we have $X \cap \mu \notin U_\mu$. Let $\text{ord}(U_\kappa) > \gamma$ if for every $X \in U_\kappa$, the set $\{\mu < \kappa : \text{ord}(U_\mu) = f_\gamma(\mu) \text{ and } X \cap \mu \in U_\mu\}$ is stationary in κ .

The definition for successor ordinals is an analogy to being Mahlo for the predecessor order.

Theorem 5.1.3. *The consistency strength of a vertical repeat point of order $\gamma \leq \kappa^+$ is strictly less than that of a measure concentrating on measurable cardinals.*

We define iterated disassociated indiscernible forcing of order up to and including κ .

For each $\gamma \leq \kappa$, let $\Delta_\gamma = \{\mu \in \Delta : \text{ord}(U_\mu) \geq f_\gamma(\mu)\}$. For each $\gamma \leq \kappa$, let

$$\mathcal{U}^\gamma = (U_\mu)_{\mu \in \Delta_\gamma}.$$

Definition 5.1.4. Let I be an interval of ordinals. The I -iterated disassociated indiscernible forcing for $\mathcal{U}$, denoted $\mathbb{D}(\mathcal{U}, I)$, has as conditions sequences $\langle s_\gamma, \mathcal{X}_\gamma \rangle_{\gamma \in I}$ such that

1. for all $\gamma \in I$, we have $s_\gamma \in [\kappa]^{<\omega}$
2. all but finitely many s_γ are empty
3. for all $\gamma \in I$, we have $\mathcal{X}_\gamma \in \mathcal{L}(\mathcal{U}^\gamma)$
4. if $\gamma \neq \gamma'$ then $s_\gamma \cap s_{\gamma'} = 0$.

A condition *extends* another, $\langle s'_\gamma, \mathcal{X}'_\gamma \rangle_{\gamma \in I} \leq \langle s_\gamma, \mathcal{X}_\gamma \rangle_{\gamma \in I}$, if for all $\gamma \in I$ we have $\langle s'_\gamma, \mathcal{X}'_\gamma \rangle \leq \langle s_\gamma, \mathcal{X}_\gamma \rangle$ in the $\mathbb{D}(\mathcal{U}^\gamma)$ order.

The definition of direct extension is analogous to that for disassociated indiscernible forcing. We sometimes use the notation $\langle \vec{s}, \vec{\mathcal{X}} \rangle$ for conditions when the index is obvious. We'll use the 'adjoin' operator $\frown$ to display the contents of $\vec{s}$, regarding the partial function as a set of pairs:

$$\langle i_0, \gamma_0 \rangle \frown \langle i_1, \gamma_1 \rangle \frown \dots \frown \langle i_n, \gamma_n \rangle.$$

Assume from now on that $\max(\Delta) = \kappa$, and that U_κ is a vertical repeat point of order $\lambda \leq \kappa$.

5.2 Basic Structure

Definition 5.2.1. Fix $\nu < \kappa$. Let $p = \langle s_\gamma, \mathcal{X}_\gamma \rangle_{\gamma \in \lambda} \in \mathbb{D}(\mathcal{U}, \lambda)$. Let

1. $p \restriction (\nu + 1) = \langle s_\gamma \cap (\nu + 1), \mathcal{X}_\gamma \cap (\nu + 1) \rangle_{\gamma < \lambda}$
2. $p \restriction \nu = \langle s_\gamma \setminus (\nu + 1), \mathcal{X}_\gamma \setminus (\nu + 1) \rangle_{\gamma < \lambda}$.
3. $\mathbb{D}_{\nu+1}(\mathcal{U}, \lambda) = \mathbb{D}((U_\mu)_{\mu \in \Delta \cap (\nu+1)}, \lambda)$.
4. $\mathbb{D}^\nu(\mathcal{U}, \lambda) = \{ \langle s_\gamma, \mathcal{X}_\gamma \rangle_{\gamma < \lambda} \in \mathbb{D}((U_\mu)_{\mu \in \Delta \setminus (\nu+1)}, \lambda) : \forall \gamma < \lambda ((s_\gamma \cup \mathcal{X}_\gamma) \cap (\nu + 1) = 0) \}$.

We do not disallow some of the long measure one entries being empty in $p \restriction (\nu + 1)$.

Proposition 5.2.2. *For any $\nu < \kappa$, we have that $\mathbb{D}(\mathcal{U}, \lambda)$ is isomorphic to the product $\mathbb{D}_{\nu+1}(\mathcal{U}, \lambda) \times \mathbb{D}^\nu(\mathcal{U}, \lambda)$ by the map $p \mapsto (p \restriction (\nu + 1), p \restriction \nu)$.*

It is interesting to note that this forcing and its conditions can be factored in two ways: ‘vertically’, as above, and although we will not utilize it, horizontally:

Definition 5.2.3. Let $p = \langle s_\gamma, \mathcal{X}_\gamma \rangle_{\gamma \in \lambda} \in \mathbb{D}(\mathcal{U}, \lambda)$. Fix $\zeta < \lambda$. Denote

1. $p \restriction \zeta = \langle s_\gamma, \mathcal{X}_\gamma \rangle_{\gamma < \zeta}$
2. $p \restriction \zeta = \langle s_\gamma, \mathcal{X}_\gamma \rangle_{\gamma \in \lambda \setminus \zeta}$.

Proposition 5.2.4. *For any $\zeta < \lambda$, we have that $\mathbb{D}(\mathcal{U}, \lambda)$ is isomorphic to the product $\mathbb{D}(\mathcal{U}, \zeta) \times \mathbb{D}(\mathcal{U}, \lambda \setminus \zeta)$ by the map $p \mapsto (p \restriction \zeta, p \restriction \zeta)$.*

5.3 Prikry Property

Definition 5.3.1. For any set K , fix some bijection $e : \kappa \rightarrow K$. Let $(X_i)_{i \in K}$ be given. Let the e -diagonal intersection be $\Delta^e X_i = \{\gamma < \kappa : \forall \nu < \gamma (\gamma \in X_{e(\nu)})\}$.

Proposition 5.3.2. *Let $(X_i)_{i \in K}$ be given such that each $X_i \in \mathcal{L}(\mathcal{U})$. Let $X = \Delta^e X_i$.*

1. *For each i we have that $X \setminus (e^{-1}(i) + 1) \subset X_i$.*
2. $X \in \mathcal{L}(\mathcal{U})$.

Definition 5.3.3. If $\vec{s}$ is a set of pairs, denote $\sup \vec{s} = \max\{i, j : \langle i, j \rangle \in \vec{s}\}$. The relation $\vec{s} \ll \vec{t}$ means that $\vec{s}, \vec{t}$ are either ordinals or finite partial functions from κ to $[\kappa]^{<\omega}$ and that $\sup(\vec{s}) < \sup(\vec{t})$.

Theorem 5.3.4. *Let σ be a formula in the language of $\mathbb{D}(\mathcal{U}, \lambda)$ and let $\langle s_\gamma, \mathcal{X}_\gamma \rangle_{\eta < \lambda}$ be any condition, where $\lambda \leq \kappa$. Then there exists a direct extension of $\langle s_\gamma, \mathcal{Z}_\eta \rangle_{\eta < \lambda}$ that decides σ .*

Proof: The proof is essentially the same as the proof for the disassociated indiscernible Prikry Property with some slight modifications to handle the iterations. Select some bijection $e : \kappa \rightarrow [\kappa \times \kappa]^{<\omega}$ such that for each cardinal ζ we have that $e^\eta \zeta = [\zeta \times \zeta]^{<\omega}$. We will use e to define our diagonal intersections. We will assume that each $s_\eta = 0$ to simplify the notation.

Let σ and $\langle 0, \vec{\mathcal{A}} \rangle$ be given.

For each $\mu \in \Delta$, $\vec{s}$, and $i < \lambda$ let $f_{\mu,i}^{\vec{s}} : \mu \rightarrow 3$ via

$$f_{\mu,i}^{\vec{s}}(\gamma) = \begin{cases} 0 & \text{if } \exists \vec{\mathcal{A}} \langle \vec{s} \frown \langle i, \gamma \rangle, \vec{\mathcal{A}} \rangle \Vdash \sigma, \\ 1 & \text{if } \exists \vec{\mathcal{A}} \langle \vec{s} \frown \langle i, \gamma \rangle, \vec{\mathcal{A}} \rangle \Vdash \neg \sigma, \\ 2 & \text{otherwise.} \end{cases}$$

Let $H_{\mu,i}^{\vec{s}} \in U_\mu$ be homogeneous for $f_{\mu,i}^{\vec{s}}$. Let $H_{\mu,i} = \Delta_{\vec{s}} H_{\mu,i}^{\vec{s}}$. For any $\vec{s}$, any $\gamma, \gamma' \in H_{\mu,i} \setminus (\sup \vec{s} + 1)$, we have that

$$\exists \vec{\mathcal{A}} \langle \vec{s} \frown \langle i, \gamma \rangle, \vec{\mathcal{A}} \rangle \parallel \sigma \text{ if and only if } \exists \vec{\mathcal{A}} \langle \vec{s} \frown \langle i, \gamma' \rangle, \vec{\mathcal{A}} \rangle \parallel \sigma,$$

and these conditions decide σ the same way.

For each $i < \lambda$, let $H_i = \cup_{\mu \in \Delta} H_{\mu,i}$. Let $\vec{\mathcal{Z}}' = (H_i \cap \mathcal{Z}_i)_{i < \lambda}$. We claim that a direct extension of $\langle 0, \vec{\mathcal{Z}}' \rangle$ decides σ . Suppose not, and we shall reach a contradiction. Fix $\langle \vec{s} \frown \langle j, \delta \rangle, \vec{\mathcal{Z}}'' \rangle \leq \langle 0, \vec{\mathcal{Z}}' \rangle$ with $|\vec{s} \frown \langle j, \delta \rangle|$ the least out of all conditions deciding σ , where $\vec{s} \ll \delta$. Without loss of generality, assume $\langle \vec{s} \frown \langle j, \delta \rangle, \vec{\mathcal{Z}}'' \rangle \Vdash \sigma$. Let μ be such that $\delta \in H_{\mu,j}$.

From now on, $\vec{s}, \mu, \delta$ and j are fixed.

For each pair $\langle i, \gamma \rangle$ where $i < \gamma < \mu$, if there is some $\vec{\mathcal{A}}$ such that $\langle \vec{s} \frown \langle i, \gamma \rangle, \vec{\mathcal{A}} \rangle$ decides σ , then let $\vec{\mathcal{A}}^{(i,\gamma)}$ be some such $\vec{\mathcal{A}}$. Let $\vec{\mathcal{A}} = (\Delta_{(i,\gamma)}^e \mathcal{X}_k^{(i,\gamma)})_{k < \lambda}$.

For each $k < \lambda$, let $Y_k = \{\nu < \mu : \{\gamma < \mu : \nu \in \mathcal{X}_k^{(j,\gamma)}\} \in U_\mu\}$ and let $H_k^* = \{\gamma < \mu : \mathcal{X}_k^{(j,\gamma)} \cap \gamma = Y_k \cap \gamma\}$. Then each $H_k^* \in U_\mu$. Let $H^* = \Delta_{k < \lambda} H_k^* \in U_\mu$. Then for any $\gamma < \mu$ we have $\mathcal{X}_k^{(j,\gamma)} \cap \gamma = Y_k \cap \gamma$ for all $k < \gamma$.

Let $\mathcal{Q}_k = ((Y_k \cup H^*) \cap \mathcal{X}_k \cap \mu) \cup (\mathcal{X}_k \setminus \mu)$. We will show that $\langle \vec{s}, \vec{\mathcal{Q}} \rangle$ forces σ . Let $\langle \vec{t}, \vec{\mathcal{R}} \rangle \leq \langle \vec{s}, \vec{\mathcal{Q}} \rangle$. Without loss of generality, $\vec{t}$ contains a pair $\langle j, \xi \rangle$ such that

$\xi \in H^*$. Consider the least such ξ . We claim $\langle \vec{t}, (\mathcal{R}_k \cap \mathcal{X}_k^{(j,\xi)})_{k < \lambda} \rangle \leq \langle \vec{s} \smallfrown \langle j, \xi \rangle, \vec{\mathcal{X}}^{(j,\xi)} \rangle$. In fact, let $\langle k, \gamma \rangle \in \vec{t}$. If $\gamma > \xi$ then $\gamma \in \mathcal{Q}_k \setminus (e^{-1}(\langle j, \xi \rangle) + 1) \subset \mathcal{X}_k^{(j,\xi)}$. Suppose $\gamma < \xi$. Since $\xi \in H^*$ and $k < \gamma < \xi$ we have that $\mathcal{X}_k^{(j,\xi)} \cap \xi = Y_k \cap \xi$, so $\langle k, \gamma \rangle \in \mathcal{X}_k^{(j,\xi)}$. Either way $\vec{t} \setminus \vec{s}$ comes from $\vec{\mathcal{X}}^{(j,\xi)}$. This establishes the claim. But $\langle \vec{s} \smallfrown \langle j, \xi \rangle, \vec{\mathcal{X}}^{(j,\xi)} \rangle$ forces σ . Hence $\langle \vec{t}, (\mathcal{R}_k \cap \mathcal{X}_k^{(j,\xi)})_{k < \lambda} \rangle$, itself an extension of an arbitrary extension of $\langle \vec{s}, \vec{\mathcal{Q}} \rangle$, forces σ . It follows that $\langle \vec{s}, \vec{\mathcal{Q}} \rangle$ forces σ . This contradicts the minimality of $|\vec{s} \smallfrown \langle j, \delta \rangle|$, finishing the proof. ■

5.4 More Structure

Theorem 5.4.1. *Fix $\nu < \kappa$. Let σ be a formula in the language of $\mathbb{D}_\nu(\mathcal{U}, \lambda) \times \mathbb{D}'(\mathcal{U}, \lambda)$. Let $(p, q) \in \mathbb{D}_\nu(\mathcal{U}, \lambda) \times \mathbb{D}'(\mathcal{U}, \lambda)$. There is a $q' \leq^* q$ such that for any $(p', q'') \leq (p, q')$ such that (p', q'') decides σ , we have that (p', q') decides σ .*

Proof: The proof is virtually the same as the proof of Theorem 3.4.4. ■

Theorem 5.4.2. *Iterated disassociated indiscernible forcing preserves cardinals.*

Proof: The obvious alteration of Theorem 3.4.5 works. ■

CHAPTER 6 GETTING MANY NORMAL MEASURES

Assume that κ is a vertical repeat point of order $\lambda \leq \kappa$. After λ -iterated disassociated indiscernible forcing, κ has exactly λ normal measures.

Definition 6.3. We define $\dot{U}_\kappa^\eta$ for $\eta < \lambda$. Let j be the ultrapower map via the normal measure U_κ . A condition $\langle s_\gamma, \mathcal{X}_\gamma \rangle_{\gamma < \lambda}$ in the generic filter forces a set $\dot{X}$ to be in $\dot{U}_\kappa^\eta$, (ie. $\langle s_\gamma, \mathcal{X}_\gamma \rangle_{\gamma < \lambda} \Vdash \dot{X} \in \dot{U}_\kappa^\eta$) iff there is a $\langle s'_\gamma, \mathcal{X}'_\gamma \rangle_{\gamma < \lambda}$ compatible with $j(\langle s_\gamma, \mathcal{X}_\gamma \rangle_{\gamma < \lambda})$ such that $s'_\eta = s_\eta \cup \{\kappa\}$, $s'_\gamma = s_\gamma$ for $\gamma \neq \eta$, and $\mathcal{X}_\gamma = \mathcal{X}'_\gamma \cap \kappa$ for all $\gamma < \lambda$.

For each $\eta < \lambda$, denote $\mathfrak{S}_\eta = \{\zeta < \kappa : \exists \langle s_\gamma, \mathcal{X}_\gamma \rangle_{\gamma \in \lambda} \in G \text{ such that } \zeta \in s_\eta\}$.

Theorem 6.4. *In the generic extension, the normal measures on κ are exactly the U_κ^η for $\eta < \lambda$.*

Proof: The proof is almost identical to those for Theorems 4.1.2 and 4.1.3, with Theorem 5.4.1 used in place of Theorem 3.4.4. ■

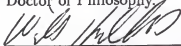
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Jeff Leaning was born in Luton England on September 1, 1971. He was raised outside of Detroit in Farmington Hills Michigan. In 1989 he moved to Gainesville Florida.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



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